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Multiple parametric resonance in Hamilton systems☆

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Abstract

The stability of a linear Hamilton system, 2π-periodic in time, with two degrees of freedom is investigated. The system depends on the parameters $\gamma_k (k = 1, 2, \ldots, s)$ and ε. The parameter ε is assumed to be small. When $\varepsilon = 0$ the system is autonomous, and the roots of its characteristic equation are equal to $\pm i\omega_1$ and $\pm i\omega_2$ (i is the square root of -1 and $\omega_1 \geq 0$, $\omega_2 \geq 0$). Cases of multiple resonance are investigated when, for certain values of $\gamma_k^{(0)}$ of the parameters γ_k, the numbers $2\omega_1$ and $2\omega_2$ are simultaneously integers. All possible cases of such resonances are considered. For small but non-zero values of ε an algorithm for constructing regions of instability in the neighbourhood of resonance values of the parameters $\gamma_k^{(0)}$ is proposed. Using this algorithm, the linear problem of the stability of the steady rotation of a dynamically symmetrical satellite when there are multiple resonances is investigated. The orbit of the centre of mass is assumed to be elliptical, the eccentricity of the orbit is small, and in the unperturbed motion the axis of symmetry of the satellite is perpendicular to the orbital plane.

Many problems on the stability and non-linear oscillations of mechanical systems lead to the need to investigate a linear Hamilton system with two or more degrees of freedom, which depend on a small parameter ε, and which is continuous and 2π-periodic in time t. Suppose the Hamilton function F of this system is analytic in ε and, when $\varepsilon = 0$, is independent of t. The Hamilton function may depend on certain parameters $\gamma_k (k = 1, 2, \ldots, s)$, and this dependence is analytic. We will also assume that the number of degrees of freedom of the system is equal to two. We will denote the frequencies of small oscillations of the system when $\varepsilon = 0$ by ω_1 and $\omega_2 (\omega_1 \geq \omega_2 \geq 0)$. They are functions of the parameters $\gamma_k (k = 1, 2, \ldots, s)$.

The problem of the stability of the system for small but non-zero values of the parameter ε (the problem of parametric resonance) has been investigated in considerable detail.[1] Formulae have been obtained for finding the boundaries of the instability regions in the first (and in many cases in the second) approximation. In the space of parameters ε, $\gamma_k (k = 1, 2, \ldots, s)$ these regions may only originate when $\varepsilon = 0$, from those points $\gamma_k = \gamma_k^{(0)}$ at which resonances of the first or second order occur, i.e. when the following relation is satisfied

$$k_1\omega_1 + k_2\omega_2 = n$$

where k_1, k_2 and n are integers, and $|k_1| + |k_2| = 1$ or 2. As a rule, single resonances have been investigated (when only one of the resonance relations are satisfied for $\gamma_k = \gamma_k^{(0)}$).

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In this paper we investigate multiple parametric resonance. It is only possible when, for $\gamma_k = \gamma_k^{(0)}$, the following two resonance relations are simultaneously satisfied

$$2\omega_1 = n_1, \quad 2\omega_2 = n_2$$

where n_1 and n_2 are non-negative integers. A constructive algorithm is proposed for obtaining the boundaries of the stability and instability regions for all possible cases of multiple resonance. Some cases of multiple resonance were considered previously in Refs. 2,3.

1. Method of investigation

We will represent the equations of the boundaries of the stability and instability regions, adjacent to the point $\gamma_k = \gamma_k^{(0)}$ when $\varepsilon = 0$, in the form of the following series

$$\gamma_k = \gamma_k^{(0)} + \sum_{m=1}^{\infty} \varepsilon^m \gamma_k^{(m)}, \quad k = 1, 2, \ldots, s \tag{1.1}$$

When solving specific problems, instead of the series (1.1) one must consider polynomials of finite degree in ε.

We substitute expansion (1.1) into the initial Hamiltonian F and expand it in series in powers of ε. The first term of this series $F_0 = F_0(q_1, q_2, p_1, p_2)$ is equal to the functions F, calculated when $\varepsilon = 0$, $\gamma_k = \gamma_k^{(0)}$. We will denote the matrix of the system of equations of motion with unperturbed Hamiltonian F_0 by $\mathbf{A}_0$.

We then make the canonical transformation $q_1, q_2, p_1, p_2 \to x_1, x_2, X_1, X_2$, specified by a constant real matrix and which reduces the unperturbed quadratic Hamiltonian F_0 to its real normal form H_0. In the new variables, the Hamiltonian of the problem can be written in the form of the following series

$$H = H_0(x_1, x_2, X_1, X_2) + \sum_{m=1}^{\infty} \frac{\varepsilon^m}{m!} H_m(x_1, x_2, X_1, X_2, t; \gamma_1^{(1)}, \ldots, \gamma_1^{(m)}, \ldots, \gamma_s^{(1)}, \ldots, \gamma_s^{(m)}) \tag{1.2}$$

where $H_m (m \geq 1)$ are quadratic forms in the canonically conjugate variables x_1, x_2, X_1 and X_2, the coefficients of which are 2π-periodic in t.

A list of all possible normal forms H_0 for the Hamiltonian F_0 can be found in Ref. 4 (see also Appendix 6 in the book[5]). Constructive algorithms for reducing the function F_0 to its normal form H_0 are also known.[6–8]

After reducing the Hamiltonian to the form (1.2) we make the close to identical τ-periodic in t canonical transformation $x_1, x_2, X_1, X_2 \to y_1, y_2, Y_1, Y_2$ ($\tau = 2\pi$ or 4π depending on the specific case of multiple resonance; see below), choosing it so that in the new Hamiltonian

$$K = H_0(y_1, y_2, Y_1, Y_2) + \sum_{m=1}^{\infty} \frac{\varepsilon^m}{m!} K_m(y_1, y_2, Y_1, Y_2, t; \gamma_1^{(1)}, \ldots, \gamma_1^{(m)}, \ldots, \gamma_s^{(1)}, \ldots, \gamma_s^{(m)}) \tag{1.3}$$

the time has been eliminated up to terms of a certain finite power n in ε (n can be fairly high). If, in expansion (1.3), we then neglect terms of higher powers than n, we arrive at an approximate linear autonomous Hamilton system with two degrees of freedom. Its characteristic equation is biquadratic and is written in the form

$$\lambda^4 + a\lambda^2 + b = 0 \tag{1.4}$$

where a and b are polynomials in ε, the coefficients of which depend on the required coefficients $\gamma_k^{(1)}, \ldots, \gamma_k^{(n)}$ ($k = 1, 2, \ldots, s$) of expansions (1.1).

Satisfaction of the following inequalities

$$a > 0, \quad b > 0, \quad d = a^2 - 4b > 0 \tag{1.5}$$

is the sufficient condition for stability. At the boundaries of the stability and instability regions the following relations are satisfied

$$a \geq 0, \quad b = 0 \quad \text{or} \quad a \geq 0, \quad d = 0 \tag{1.6}$$

Equating the coefficients of $\varepsilon, \varepsilon^2, \ldots, \varepsilon^n$ on the left-hand sides of these relations to zero, we obtain a system of equations for the coefficients $\gamma_k^{(1)}, \ldots, \gamma_k^{(n)}$ $(k=1, 2, \ldots, s)$ of expansions (1.1).

In certain resonance cases the biquadratic Eq. (1.4) can be split into two quadratic equations, and it becomes easier to analyse them (see below, Sections 3.2, 3.4 and 5).

To construct the replacement of variables $x_1, x_2, X_1, X_2 \to y_1, y_2, Y_1, Y_2$ and the transformed Hamiltonian (1.3) we will use a modification of the Deprit-Hori method, described in Ref. 9. We will write the generating function W of the Lie transformation, employed in the Deprit-Hori method, in the form of a series

$$W = \sum_{m=0}^{\infty} \frac{\varepsilon^m}{m!} W_{m+1}(y_1, y_2, Y_1, Y_2, t)$$

We have the following recurrence relations for calculating the converted Hamiltonian[6,9]

$$\begin{aligned} K_m &= H_m + \sum_{j=1}^{m-1} (C_{m-1}^{j-1} L_j H_{m-j} - C_{m-1}^{j} K_{j, m-j}) + L_m H_0 - \frac{\partial W_m}{\partial t} \\ K_{j,i} &= L_j K_i - \sum_{s=1}^{j-1} C_{j-1}^{s-1} L_s K_{j-s,i}, \quad C_r^k = \frac{r!}{k!(r-k)!} \end{aligned} \tag{1.7}$$

Here we have denoted the Poisson bracket of the functions f and W_i by $L_i f$

$$L_i f = \sum_{l=1}^{2} \left(\frac{\partial f}{\partial y_l} \frac{\partial W_i}{\partial Y_l} - \frac{\partial f}{\partial Y_l} \frac{\partial W_i}{\partial y_l} \right)$$

In the following sections, the algorithm for obtaining the boundaries of the stability and instability regions is considered in more detail as it applies to each of the possible multiple resonances. It should be noted that, in specific problems, the algorithm may turn out to be quite cumbersome, and hence, as a rule, its realization requires the use of Analytical Computation Systems.

2. The resonance $\omega_1 = \omega_2 = 0$

We will first consider the case when both frequencies of small oscillations are equal to zero when $\gamma_k = \gamma_k^{(0)}$. This case is encountered fairly rarely in practice. However, as will be seen from what follows, from the computational point of view many other special cases of multiple resonance reduce to it.

Depending on the value of the rank r of the matrix $\mathbf{A}_0$ of the equations of motion of the unperturbed system, when $\omega_1 = \omega_2 = 0$ we must distinguish four cases ($r = 3$, 2, 1 or 0), which do not reduce to one another. We will consider them in sequence.

2.1. *The case r = 3*

In this case the function H_0 in the converted Hamiltonian (1.3) has the form[7]

$$H_0 = \frac{1}{2}\delta Y_1^2 - y_1 y_2 \quad (\delta = \pm 1) \tag{2.1}$$

According to relations (1.7), finding the first approximation in ε reduces to considering the following linear partial differential equation

$$K_1 = H_1(y_1, y_2, Y_1, Y_2, t; \gamma_1^{(1)}, \ldots, \gamma_s^{(1)}) + (H_0, W_1) - \frac{\partial W_1}{\partial t} \tag{2.2}$$

We will write the quadratic form H_1 in this equation in the form

$$H_1 = \sum h_{m_1, m_2, n_1, n_2}^{(1)} y_1^{m_1} y_2^{m_2} Y_1^{n_1} Y_2^{n_2}$$

where the summation is carried out over the non-negative integers m_1, m_2, n_1 and n_2, the sum of which is equal to two. We will also represent the desired quadratic forms K_1 and W_1 in the form of similar sums.

It is required to choose the coefficients $w^{(1)}_{m_1,m_2,n_1,n_2}$ of the form W_1 so that they are 2π-periodic in t, while the coefficients $k^{(1)}_{m_1,m_2,n_1,n_2}$ of the form K_1 are constant.

Equating the coefficients of like powers of y_1, y_2, Y_1 and Y_2 on the left and right sides of equality (2.2), we obtain the following ten relations

$$\begin{aligned}
&\frac{dw^{(1)}_{0002}}{dt} = h^{(1)}_{0002} - k^{(1)}_{0002}, \quad \frac{dw^{(1)}_{1001}}{dt} = h^{(1)}_{1001} - 2w^{(1)}_{0002} - k^{(1)}_{1001}\\
&\frac{dw^{(1)}_{0011}}{dt} = h^{(1)}_{0011} - \delta w^{(1)}_{1001} - k^{(1)}_{0011}, \quad \frac{dw^{(1)}_{0101}}{dt} = h^{(1)}_{0101} - w^{(1)}_{0011} - k^{(1)}_{0101}\\
&\frac{dw^{(1)}_{2000}}{dt} = h^{(1)}_{2000} - w^{(1)}_{1001} - k^{(1)}_{2000}, \quad \frac{dw^{(1)}_{1010}}{dt} = h^{(1)}_{1010} - 2\delta w^{(1)}_{2000} - w^{(1)}_{0011} - k^{(1)}_{1010}\\
&\frac{dw^{(1)}_{1100}}{dt} = h^{(1)}_{1100} - w^{(1)}_{1010} - w^{(1)}_{0101} - k^{(1)}_{1100}, \quad \frac{dw^{(1)}_{0020}}{dt} = h^{(1)}_{0020} - \delta w^{(1)}_{1010} - k^{(1)}_{0020}\\
&\frac{dw^{(1)}_{0110}}{dt} = h^{(1)}_{0110} - \delta w^{(1)}_{1100} - 2w^{(1)}_{0020} - k^{(1)}_{0110}, \quad \frac{dw^{(1)}_{0200}}{dt} = h^{(1)}_{0200} - w^{(1)}_{0110} - k^{(1)}_{0200}
\end{aligned} \tag{2.3}$$

Considering these relations in succession, we can obtain the functions $w^{(1)}_{m_1,m_2,n_1,n_2}(t)$ and $k^{(1)}_{m_1,m_2,n_1,n_2}$, which satisfy the requirements formulated above. We obtain

$$\begin{aligned}
&k^{(1)}_{0002} = \frac{1}{2\pi}\int_0^{2\pi} h^{(1)}_{0002}dt, \quad w^{(1)}_{0002} = \int(h^{(1)}_{0002} - k^{(1)}_{0002})dt\\
&k^{(1)}_{1001} = \frac{1}{2\pi}\int_0^{2\pi} (h^{(1)}_{1001} - 2w^{(1)}_{0002})dt, \quad w^{(1)}_{1001} = \int(h^{(1)}_{1001} - 2w^{(1)}_{0002} - k^{(1)}_{1001})dt\\
&k^{(1)}_{0011} = \frac{1}{2\pi}\int_0^{2\pi} (h^{(1)}_{0011} - \delta w^{(1)}_{1001})dt, \quad w^{(1)}_{0011} = \int(h^{(1)}_{0011} - \delta w^{(1)}_{1001} - k^{(1)}_{0011})dt
\end{aligned} \tag{2.4}$$

and so on.

In exactly the same way we can obtain second and higher approximations from relations (1.7) and (2.1). In the limit, we obtain the transformed Hamiltonian (1.3) in the form

$$K = H_0 + \sum k_{m_1,m_2,n_1,n_2} y_1^{m_1} y_2^{m_2} Y_1^{n_1} Y_2^{n_2} \tag{2.5}$$

The function H_0 is defined by (2.1), the summation is carried out over the non-negative integers m_1, m_2, n_1 and n_2, the sum of which is equal to 2, while the constant coefficients k_{m_1,m_2,n_1,n_2} are calculated from the formula

$$k_{m_1,m_2,n_1,n_2} = \sum_{m=1}^{\infty} \frac{\varepsilon^m}{m!} k^{(m)}_{m_1,m_2,n_1,n_2}(\gamma^{(1)}_1, \ldots, \gamma^{(m)}_1, \ldots, \gamma^{(1)}_s, \ldots, \gamma^{(m)}_s) \tag{2.6}$$

where $k^{(m)}_{m_1,m_2,n_1,n_2}$ is the coefficient of $y_1^{m_1} y_2^{m_2} Y_1^{n_1} Y_2^{n_2}$ in the quadratic form K_m.

Confining ourselves in the expansions (2.5) to terms, the power of which in ε does not exceed n, we can obtain relations from conditions (1.6) which the quantities $\gamma^{(1)}_k, \ldots, \gamma^{(n)}_k (k = 1, 2, \ldots, s)$ must satisfy, which define the boundaries of the stability and instability regions (1.1).

2.2. The case r = 2

Technically, the investigation of this case is similar to the investigation of the case $r = 3$ carried out above, and hence we will merely write the equations which define the first approximation. For $r = 2$ we have[7]

$$H_0 = \frac{1}{2}\delta_1 Y_1^2 + \frac{1}{2}\delta_2 Y_2^2; \quad \delta_1 = \pm 1, \quad \delta_2 = \pm 1 \tag{2.7}$$

and from Eq. (2.2) we obtain ten of these relations

$$\begin{aligned}
&\frac{dw_{2000}^{(1)}}{dt} = h_{2000}^{(1)} - k_{2000}^{(1)}, \quad \frac{dw_{1100}^{(1)}}{dt} = h_{1100}^{(1)} - k_{1100}^{(1)} \\
&\frac{dw_{1010}^{(1)}}{dt} = h_{1010}^{(1)} - 2\delta_1 w_{2000}^{(1)} - k_{1010}^{(1)}, \quad \frac{dw_{1001}^{(1)}}{dt} = h_{1001}^{(1)} - \delta_2 w_{1100}^{(1)} - k_{1001}^{(1)} \\
&\frac{dw_{0200}^{(1)}}{dt} = h_{0200}^{(1)} - k_{0200}^{(1)}, \quad \frac{dw_{0110}^{(1)}}{dt} = h_{0110}^{(1)} - \delta_1 w_{1100}^{(1)} - k_{0110}^{(1)} \\
&\frac{dw_{0101}^{(1)}}{dt} = h_{0101}^{(1)} - 2\delta_2 w_{0200}^{(1)} - k_{0101}^{(1)}, \quad \frac{dw_{0020}^{(1)}}{dt} = h_{0020}^{(1)} - \delta_1 w_{1010}^{(1)} - k_{0020}^{(1)} \\
&\frac{dw_{0011}^{(1)}}{dt} = h_{0011}^{(1)} - \delta_1 w_{1001}^{(1)} - \delta_2 w_{0110}^{(1)} - k_{0011}^{(1)}, \quad \frac{dw_{0002}^{(1)}}{dt} = h_{0002}^{(1)} - \delta_2 w_{0101}^{(1)} - k_{0002}^{(1)}
\end{aligned} \tag{2.8}$$

This system of equations is considered in the same way as system (2.3). We obtain the following expression for the coefficients of the quadratic forms K_1 and W_1

$$\begin{aligned}
&k_{2000}^{(1)} = \frac{1}{2\pi}\int_0^{2\pi} h_{2000}^{(1)}(dt), \quad w_{2000}^{(1)} = \int (h_{2000}^{(1)} - k_{2000}^{(1)})dt \\
&k_{1100}^{(1)} = \frac{1}{2\pi}\int_0^{2\pi} h_{1100}^{(1)}dt, \quad w_{1100}^{(1)} = \int (h_{1100}^{(1)} - k_{1100}^{(1)})dt \\
&k_{1010}^{(1)} = \frac{1}{2\pi}\int_0^{2\pi} (h_{1010}^{(1)} - 2\delta_1 w_{2000}^{(1)})dt, \quad w_{1010}^{(1)} = \int (h_{1010}^{(1)} - 2\delta_1 w_{2000}^{(1)} - k_{1010}^{(1)})dt
\end{aligned} \tag{2.9}$$

and so on.

2.3. The case r = 1

If $r = 1$, the function H_0 in (1.3) is given by the equality[7]

$$H_0 = \frac{1}{2}\delta_2 Y_2^2; \quad \delta_2 = \pm 1 \tag{2.10}$$

The investigation of this case hardly differs from that of the case $r = 2$. We merely need to put $\delta_1 = 0$ in the equations of the first approximation (and in the equations of subsequent approximations, obtained from relation (1.7)), and replace the function (2.7) by the function (2.10) in the converted Hamiltonian (2.5).

2.4. The case r = 0

In this case $H_0 = 0$. Eq. (2.2) of the first approximation has the form

$$K_1 = H_1(y_1, y_2, Y_1, Y_2, t; \mu_1^{(1)}, \ldots, \mu_s^{(1)}) - \frac{\partial W_1}{\partial t} \tag{2.11}$$

and its solution can be found very simply. The quadratic form K_1 with constant coefficients and the 2π-periodic quadratic form W_1 can be taken as follows:

$$K_1 = \frac{1}{2\pi}\int_0^{2\pi} H_1 dt, \quad W_1 = \int (H_1 - K_1)dt \tag{2.12}$$

We can similarly construct solutions of the equations of the second and higher approximations.

3. The resonance $2\omega_1 = n_1$, $\omega_2 = 0$

We will consider multiple resonances when the frequency ω_2 of small oscillations of the unperturbed system is equal to zero, while the other frequency ω_1 is non-zero but $2\omega_1 = n_1$, where n_1 is a natural number. Here we must distinguish four cases depending on the rank r of the matrix $\mathbf{A}_0$ of the equations of motion with unperturbed Hamiltonian F_0 (it can be equal to three or two) and also depending on whether n_1 is even or odd.

3.1. The case r = 3 and n_1 is an even number

In the Hamilton function (1.2) we have

$$H_0 = \frac{1}{2}\sigma_1(x_1^2 + X_1^2) + \frac{1}{2}\delta_2 X_2^2; \quad \sigma_1 = \delta_1\omega_1, \quad \delta_1 = \pm 1, \quad \delta_2 = \pm 1 \tag{3.1}$$

Before converting the Hamilton function (1.2) to the form (1.3) by the Deprit-Hori method we will make the canonical univalent transformation $x_1, x_2, X_1, X_2 \to x_1', x_2', X_1', X_2'$ using the following formulae

$$\begin{aligned} x_1 &= \cos(\sigma_1 t)x_1' + \sin(\sigma_1 t)X_1', \quad x_2 = x_2' \\ X_1 &= -\sin(\sigma_1 t)x_1' + \cos(\sigma_1 t)X_1', \quad X_2 = X_2' \end{aligned} \tag{3.2}$$

This replacement of variables eliminates the term $\frac{1}{2}\sigma_1(x_1^2 + X_1^2)$ from the Hamiltonian (3.1). If we omit the primes in the notation of the new variables, the new Hamilton function can be written in the form (1.2), where $H_0 = \frac{1}{2}\delta_2 X_2^2$, while the functions H_m are 2π-periodic in t, as before. Therefore, the case of multiple resonance being investigated has also been reduced to one of the cases of two zero frequencies, considered in Section 2.3.

3.2. The case r = 3 and n_1 is odd

As in the preceding case, the function H_0 is given by Eq. (3.1), and after the replacement of variables (3.2) the Hamiltonian can be written in the form (1.2), where $H_0 = \frac{1}{2}\delta_2 X_2^2$. Hence, as in the previous case, we arrive at the case of resonance at two zero frequencies, considered in Section 2.3. However, unlike the previous case, when n_1 is even, here, for odd n_1, the functions H_m, after replacement (3.2), will generally speaking have a period of 4π in t and not 2π. When finding the constant coefficients of the quadratic forms K_m using formulae of the form (2.8) (in which, as in Section 2.3, we must put $\delta_1 = 0$), the mean values of the corresponding functions must be calculated with a period of 4π with respect to time.

The converted Hamiltonian (2.5) will not contain all ten but only six monomials

$$K = k_{2000}y_1^2 + k_{1010}y_1Y_1 + k_{0020}Y_1^2 + k_{0200}y_2^2 + k_{0101}y_2Y_2 + \frac{1}{2}(\delta_2 + 2k_{0002}Y_2^2) \tag{3.3}$$

In a system with such a Hamiltonian the equations for the variables y_1, Y_1 and y_2, Y_2 can be split. We put

$$d_1 = k_{1010}^2 - 4k_{2000}k_{0020}, \quad d_2 = k_{0101}^2 - 2k_{0200}(\delta_2 + 2k_{0002}) \tag{3.4}$$

When at least one of the inequalities $d_1 > 0$ or $d_2 > 0$ is satisfied we will have instability. At the boundaries of the stability and instability regions at least one of the quantities d_1 or d_2 vanishes. Equating the coefficients of the expansions of the functions (3.4) in series in powers of ε to zero, we obtain relations determining the expansion coefficients (1.1).

3.3. The case $r = 2$ and n_1 even

The function H_0 in expression (1.2) has the form

$$H_0 = \frac{1}{2}\sigma_1(x_1^2 + X_1^2); \quad \sigma_1 = \delta_1\omega_1, \quad \delta_1 = \pm 1 \tag{3.5}$$

After the replacement of variables (3.2) we arrive at the Hamiltonian (1.2) in which $H_0 = 0$. Hence, the case of multiple resonance here also reduces to one of the cases of resonance $\omega_1 = \omega_2 = 0$, considered in Section 2.4.

3.4. The case $r = 2$ and n_1 odd

The function H_0, as in the previous case, is specified by equality (3.5), and after making the replacement (3.2) we again arrive at the case $H_0 = 0$ from Section 2.4. The functions H_m in (1.2) will, generally speaking, have a period of 4π. When finding the quadratic forms K_m using formulae of the form (2.12), the mean values are calculated for a period of 4π.

The converted Hamiltonian (2.5) will only contain six monomials and is given by equality (3.3), in which the quantity δ_2 must be put equal to zero.

As in the case (3.2), the boundaries of the stability and instability regions are given by the equalities $d_1 = 0$ or $d_2 = 0$, where d_1 and d_2 are the functions (3.4) in which $\delta_2 = 0$.

4. The resonance $2\omega_1 = 2\omega_2 = n$

We will now consider possible multiple resonances, when not one of the frequencies of small oscillations of the unperturbed system is equal to zero. In this section the frequencies are assumed to be equal: $\omega_1 = \omega_2 = \omega$, where $2\omega = n$, and n is a natural number. Depending on the rank r of the matrix $(\mathbf{A}_0 - i\omega\mathbf{E})$ (it can be equal to three or two) here two cases are possible which do not reduce to one another.

4.1. The case $r = 3$

In this case we have the following expression for the function H_0 in (1.2)

$$H_0 = \frac{1}{2}\delta(X_1^2 + X_2^2) + \omega(x_1X_2 - x_2X_1); \quad \delta = \pm 1 \tag{4.1}$$

Parametric resonance in a system with Hamiltonian (4.1) was investigated previously in Ref. 3. If we introduce new canonically conjugate variables x'_j, $X'_j (j = 1, 2)$ using the univalent canonical transformation

$$\begin{aligned} x_1 &= \cos(\omega t)x'_1 - \sin(\omega t)x'_2, \quad x_2 = \sin(\omega t)x'_1 + \cos(\omega t)x'_2 \\ X_1 &= \cos(\omega t)X'_1 - \sin(\omega t)X'_2, \quad X_2 = \sin(\omega t)X'_1 + \cos(\omega t)X'_2 \end{aligned} \tag{4.2}$$

the term $\omega(x_1X_2 - x_2X_1)$ in Hamiltonian (4.1) will be cancelled. In the new variables, the Hamilton function (1.2) remains 2π-periodic in t, and its unperturbed part has the form (we will omit the primes in the notation of the new variables)

$$H_0 = \frac{1}{2}\delta(X_1^2 + X_2^2); \quad \delta = \pm 1 \tag{4.3}$$

Hence, we again arrive at one of the cases of two zero frequencies considered in Section 3.3. We need only put $\delta_1 = \delta_2 = \delta$ when carrying out calculations using formulae of the form (2.8) and (2.9).

4.2. The case $r = 2$

The function H_0 in expression (1.2) is the sum of the Hamiltonians of two harmonic oscillators with the same frequencies ω

$$H_0 = \frac{1}{2}\sigma_1(x_1^2 + X_1^2) + \frac{1}{2}\sigma_2(x_2^2 + X_2^2); \quad \sigma_1 = \delta_1\omega, \quad \sigma_2 = \delta_2\omega, \quad \delta_1 = \pm 1, \quad \delta_2 = \pm 1 \tag{4.4}$$

If we make the univalent canonical replacement of variables $x_j, X_j \to x'_j, X'_j (j = 1, 2)$ using the formulae

$$x_j = \cos(\sigma_j t)x'_j + \sin(\sigma_j t)X'_j, \quad X_j = -\sin(\sigma_j t)x'_j + \cos(\sigma_j t)X'_j; \quad j = 1, 2 \tag{4.5}$$

we obtain the Hamiltonian (1.2), 2π-periodic in t, in which $H_0 = 0$, i.e. the resonance case being investigated reduces to the case of two zero frequencies considered in Section 2.4.

5. The resonance $2\omega_1 = n_1$, $2\omega_2 = n_2$ ($n_1 \neq 0, n_2 \neq 0$)

It remains to consider multiple resonances, when the frequencies of the unperturbed system are non-zero and different. Such resonances were investigated previously in Ref. 2. The normal form of the unperturbed Hamiltonian F_0 is given by equality (4.4) in which $\sigma_1 = \delta_1\omega_1$, $\sigma_2 = \delta_2\omega_2$, $\delta_1 = \pm 1$, $\delta_2 = \pm 1$. After replacing the variables (4.5) we arrive at Hamiltonian (1.2) in which $H_0 = 0$. Hence, the further investigation is carried as in Section 2.4.

It is useful to distinguish two cases in the calculations. When both numbers n_1 and n_2 are even or they are both odd, the converted Hamiltonian K (given by Eq. (2.5) in which $H_0 = 0$), will contain, generally speaking, all ten monomials and the boundaries of the stability and instability regions will be found from relations (1.6). If one of the numbers n_1 or n_2 is even, while the other is odd, the Hamiltonian K will not contain all ten but only six monomials and is given by an equality of the form (3.3), in which $\delta_2 = 0$. The boundaries of the stability and instability regions are found from the equalities $d_1 = 0$ and $d_2 = 0$, where d_1 and d_2 are the quantities (3.4) in which $\delta_2 = 0$.

6. The stability of the steady rotation of a dynamically symmetrical satellite in an elliptic orbit

Suppose the centre of mass of the satellite moves in an elliptic orbit of eccentricity e in a central Newtonian gravitational field. The satellite is a rigid body, the central ellipsoid of which is a spheroid. The equatorial and polar moments of inertia of the satellite will be denoted by A and C. It is well known,[10] that the problem of the motion of a satellite about a centre of mass under the action of gravitational moments allows of a particular solution, for which the axis of dynamic symmetry of the satellite is perpendicular to the orbital plane, while the satellite itself rotates around the axis of symmetry with a constant angular velocity r_0.

The Hamilton function F, corresponding to the linearized equations of perturbed motion of the axis of symmetry in the neighbourhood of the normal to the orbital plane, has the form[11]

$$\begin{aligned} F = {} & \frac{1}{2}\left[\frac{\alpha^2\beta^2(1-e^2)^3}{(1+e\cos\nu)^2} - \alpha\beta(1-e^2)^{3/2} + 3(\alpha-1)(1+e\cos\nu)\right]q_1^2 + \\ & + \left[\frac{\alpha\beta(1-e^2)^{3/2}}{(1+e\cos\nu)^2} - 1\right]q_1p_2 + \frac{1}{2}\alpha\beta(1-e^2)^{3/2}q_2^2 + q_2p_1 + \frac{1}{2(1+e\cos\nu)^2}(p_1^2+p_2^2) \end{aligned} \tag{6.1}$$

Here $\alpha = C/A$, $\beta = r_0/\omega_0$ ($0 < \alpha \le 2$, $-\infty < \beta < \infty$) and ω_0 is the average motion of the centre of mass in the orbit and we have taken the true anomaly ν as the independent variable.

If we introduce the notation

$$s_1 = \alpha\beta - 1, \quad s_2 = \alpha\beta + 3\alpha - 4, \quad s_3 = \alpha^2\beta^2 - 2\alpha\beta + 3\alpha - 1 \tag{6.2}$$

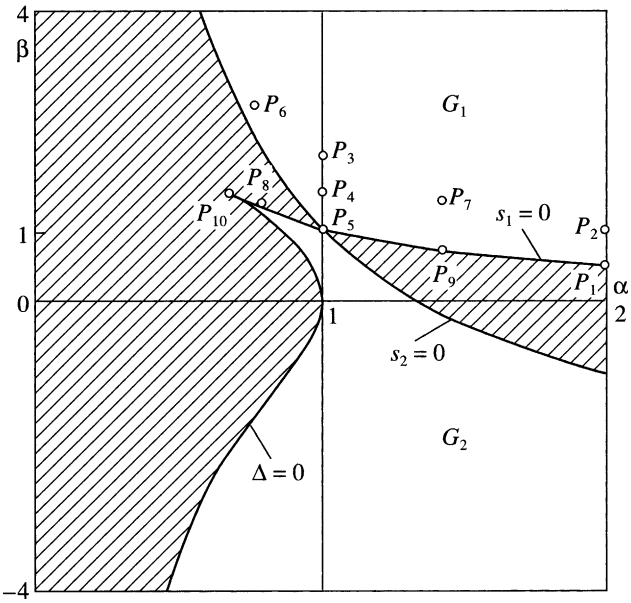

Fig. 1.

then, in a circular orbit ($e = 0$) the regions G_1 and G_2 (Fig. 1) of stability in the first approximation of the steady rotation of the satellite can be specified by the inequalities[12]

$$\dot{s}_1 > 0, \quad s_2 > 0 \quad \text{and} \quad s_1 < 0, \quad s_2 < 0, \quad \Delta > 0 \tag{6.3}$$

respectively. Here

$$\Delta = s_3^2 - 4s_1s_2 = \alpha^4\beta^4 - 4\alpha^3\beta^3 + 6\alpha^3\beta^2 - 2\alpha^2\beta^2 - 24\alpha^2\beta + 24\alpha\beta + 9\alpha^2 + 6\alpha - 15 \tag{6.4}$$

The instability regions are shown hatched in Fig. 1. For values of the parameters α and β, belonging to the regions G_1 and G_2 or their boundaries, the frequencies ω_1 and $\omega_2(\omega_1 \geq \omega_2 \geq 0)$ of small oscillations of the axis of symmetry in the neighbourhood of the normal to the orbital plane are the roots of the equation

$$\omega^4 - s_3\dot{\omega}^2 + s_1s_2 = 0 \tag{6.5}$$

A denumerable set of curves exists in regions G_1 and G_2,[11] on which first-order and second-order resonances occur. For small values of the eccentricity e many multiple parametric resonances are possible. Below we will investigate the stability of the steady rotation of a satellite for some of these resonances only. For small e we obtain stability and instability regions in the neighbourhood of ten points $P_i(\alpha, \beta)$ (see Fig. 1). The cases investigated, on the one hand, are of independent interest, while on the other they illustrate the majority of possible multiple parametric resonances considered in a Hamilton system with two degrees of freedom, 2π-periodic in the independent variable. Other examples of multiple resonances in the problem of the stability of the motion of a dynamically symmetrical satellite in an elliptic orbit, not considered in this paper, were investigated previously in Refs. 3,13.

6.1. A satellite, the geometry of the masses of which corresponds to a plate ($2A = C$)

In the case of a plate, $\alpha = 2$ and Hamiltonian (6.1) depends on two parameters β and e. We will consider the possibility of the existence of multiple parametric resonance in an orbit with a small eccentricity for values of the parameter β not belonging to the interval $-1 < \beta < 1/2$ of instability of the steady rotation of the satellite in a circular orbit (Fig. 1).

In Fig. 2 we show the frequencies ω_1 and ω_2 as a function of the parameter β in the case of a circular orbit. For all the values of β considered the inequalities $\omega_1 > 1.94596$ and $0 \leq \omega_2 < 1.25996$ hold. Only two cases of multiple resonance are possible: $\omega_1 = 2$ and $\omega_2 = 0$ (when $\beta = ½$, see point P_1 in Fig. 1) and $\omega_1 = 2$ and $\omega_2 = 1$ (when $\beta = 1$, see point P_2 in Fig. 1).

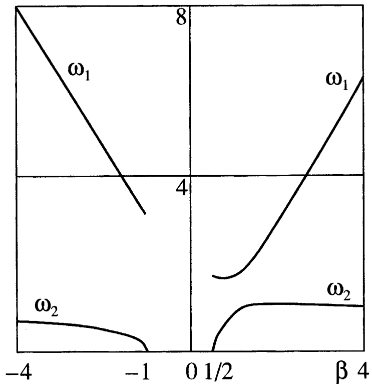

Fig. 2.

The resonance $\omega_1 = 2$ and $\omega_2 = 0$. In Hamiltonian (6.1) we put

$$\beta = \frac{1}{2} + e\mu_1 + e^2\mu_2 + e^3\mu_3 + e^4\mu_4 + \ldots$$

and we expand it in series in powers of e. The unperturbed Hamiltonian F_0 has the form

$$F_0 = \frac{3}{2}q_1^2 + \frac{1}{2}q_2^2 + q_2p_1 + \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$$

The univalent canonical transformation

$$q_1 = -\frac{\sqrt{2}}{2}X_1 + \frac{\sqrt{3}}{6}X_2, \quad q_2 = \frac{\sqrt{2}}{4}x_1 + \frac{\sqrt{3}}{2}x_2, \quad p_1 = \frac{3\sqrt{2}}{4}x_1 - \frac{\sqrt{3}}{2}x_2, \quad p_2 = \frac{\sqrt{2}}{2}X_1 + \frac{\sqrt{3}}{2}X_2$$

reduces the function F_0 to its real normal form H_0,

$$H_0 = x_1^2 + X_1^2 + \frac{1}{2}X_2^2$$

Further, using the algorithm in Section 3.1, we can obtain the Hamiltonian (2.5). Up to first powers in e it will take the form

$$K = \frac{1}{2}Y_2^2 + e\left(-\frac{3}{16}y_1^2 + \frac{3}{4}y_2^2 - \frac{3}{16}Y_1^2 + \frac{\sqrt{6}}{8}Y_1Y_2 + \frac{7}{12}Y_2^2\right)\mu_1 \tag{6.6}$$

The coefficients a and b of characteristic Eq. (1.4) of the approximate system with Hamiltonian (6.6) have the form

$$a = \frac{3}{2}e\mu_1 + O(e^2), \quad b = \frac{27}{128}e^3\mu_1^3 + O(e^4)$$

Hence, by relations (1.6), it follows that at the boundaries of the stability and instability regions the equality $\mu_1 = 0$ must be satisfied.

Putting $\mu_1 = 0$ and calculating the Hamiltonian (6.6) up to terms of the fourth power in e inclusive, we obtain the following expressions for the coefficients a and b and for the quantity d, which occurs in the last inequality of (1.5)

$$a = \frac{3}{2}e^2\mu_2 + O(e^3), \quad b = b^{(6)}e^6 + b^{(7)}e^7 + b^{(8)}e^8 + O(e^9), \quad d = \frac{9}{4}e^4\mu_2^2 + O(e^4)$$

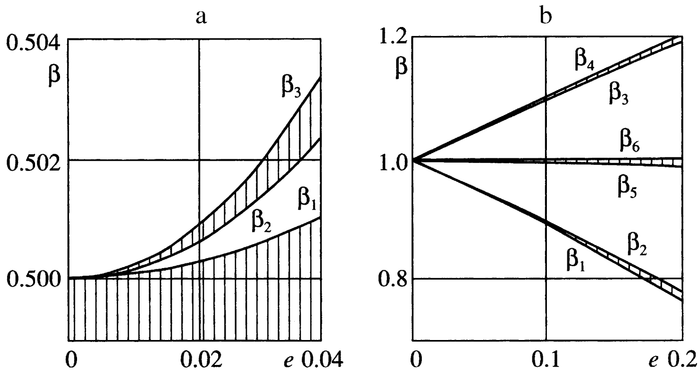

Fig. 3.

where

$$b^{(6)} = \frac{27}{128}\mu_2^3 - \frac{27}{32}\mu_2^2 + \frac{4185}{4096}\mu_2 - \frac{729}{2048}, \quad b^{(7)} = \left(\frac{81}{128}\mu_2^2 - \frac{27}{16}\mu_2 + \frac{4185}{4096}\right)\mu_3$$

$$b^{(8)} = \left(\frac{81}{128}\mu_2^2 - \frac{27}{16}\mu_2 + \frac{4185}{4096}\right)\mu_4 + \left(\frac{81}{128}\mu_2 - \frac{27}{32}\right)\mu_3^2 -$$
$$- \frac{873}{2048}\mu_2^4 - \frac{2637}{1024}\mu_2^3 + \frac{1030995}{131072}\mu_2^2 - \frac{256365}{65536}\mu_2 + \frac{167427}{8388608}$$

For $\mu_2 = 0$ and small e, the value of b is negative, and hence it follows from relations (1.6), that the boundaries of the stability and instability regions are found from the conditions $a \geq 0$ and $b = 0$. Equating the quantity $b^{(6)}$ to zero, we obtain three positive values of the quantity μ_2, corresponding to the required boundaries: $1 \pm \sqrt{10}/8$ and 2. For these values of μ_2, from the equation $b^{(7)} = 0$ we obtain $\mu_3 = 0$. Then the equation $b^{(8)} = 0$ for $\mu_3 = 0$ and the three values of μ_2 obtained gives three values of μ_4, corresponding to the boundaries of the stability and instability regions. Hence, we obtain, with an error of the order of e^5, the following three equations of the boundary curves issuing from the point (0, ½) in the e, β plane

$$\beta_i(e) = \frac{1}{2} + \left[1 + (-1)^i \frac{\sqrt{10}}{8}\right]e^2 + \left[\frac{79567}{12288} + (-1)^i \frac{5891}{3072}\sqrt{10}\right]e^4, \quad i = 1, 2$$
$$\beta_3(e) = \frac{1}{2} + 2e^2 + \frac{129871}{6144}e^4 \tag{6.7}$$

Curves (6.7) are shown in Fig. 3a. The instability regions are shown hatched.

The resonance $\omega_1 = 2$ and $\omega_2 = 1$. We put

$$\beta = 1 + e\mu_1 + e^2\mu_2 + e^3\mu_3 + e^4\mu_4 + \dots \tag{6.8}$$

and we expand Hamiltonian (6.1) in series in powers of e. For the unperturbed Hamiltonian F_0 we have the expression

$$F_0 = \frac{5}{2}q_1^2 + q_1 p_2 + q_2^2 + q_2 p_1 + \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$$

In the variables x_1, x_2, X_1 and X_2, which occur in the canonical transformation

$$q_1 = \frac{\sqrt{2}}{2}x_1, \quad q_2 = x_2, \quad p_1 = -x_2 + \sqrt{2}X_1, \quad p_2 = -\frac{\sqrt{2}}{2}x_1 + X_2$$

the function F_0 takes its normal form

$$H_0 = x_1^2 + X_1^2 + \frac{1}{2}(x_2^2 + X_2^2)$$

By the algorithm of Section 5, we can eliminate from Hamiltonian (1.2) its unperturbed part H_0 and obtain the transformed Hamiltonian (1.3). The coefficients a and b of characteristic Eq. (1.4) and the quantity d from (1.5) can be represented in the form of series in powers of e

$$a = \sum_{k=2}^{\infty} a^{(k)} e^k, \quad b = \sum_{k=4}^{\infty} b^{(k)} e^k, \quad d = \sum_{k=4}^{\infty} d^{(k)} e^k \tag{6.9}$$

where $a^{(k)}$, $b^{(k)}$, $d^{(k)}$ are functions of the coefficients of the expansion (6.8), and

$$a^{(2)} = \frac{1}{4}(5\mu_1^2 + 9), \quad b^{(4)} = \frac{1}{64}(4\mu_1^2 - 9)^2, \quad d^{(4)} = \frac{9}{16}\mu_1^2(\mu_1^2 + 18) \tag{6.10}$$

Since the coefficient a is positive for small e, according to relations (1.6), the boundaries of the stability and instability regions are defined by the equalities $b=0$ and $d=0$.

Consider the equality $b=0$. It can be seen from formulae (6.10) that the coefficient $b^{(4)}$ in the expansion of b in series vanishes if $\mu_1 = \pm 3/2$. Calculations show that in that case the coefficient $b^{(5)}$ also vanishes, while the coefficients $b^{(6)}$, $b^{(7)}$ and $b^{(8)}$ will be as follows:

$$b^{(6)} = \frac{9}{102400}(160\mu_2 + 57)(160\mu_2 - 3)$$

$$b^{(7)} = \frac{9}{320}(160\mu_2 + 27)\mu_3 \pm \left(\frac{3}{2}\mu_2^3 + \frac{3213}{320}\mu_2^2 - \frac{88209}{25600}\mu_2 - \frac{3420657}{4096000}\right)$$

$$b^{(8)} = \frac{9}{320}(160\mu_2 + 27)\mu_4 + \frac{9}{4}\mu_3^2 \pm \left(\frac{9}{2}\mu_2^2 + \frac{3213}{160}\mu_2 - \frac{88209}{25600}\right)\mu_3 +$$
$$+ \frac{1}{4}\mu_2^4 + \frac{2007}{160}\mu_2^3 - \frac{227097}{12800}\mu_2^2 + \frac{21477969}{1024000}\mu_2 + \frac{5795896401}{655360000}$$

Equating the quantities $b^{(6)}$, $b^{(7)}$, $b^{(8)}$ to zero, we obtain a system of equations for finding the quantities μ_2, μ_3 and μ_4, defining the boundaries of the stability and instability regions (on which $b=0$) apart from terms of the fourth power in e inclusive. Calculations showed that four boundaries $\beta = \beta_i(e)$ $(i=1, 2, 3, 4)$ exist, where $\beta_3(e)=\beta_1(-e)$, $\beta_4(e)=\beta_2(-e)$, while

$$\beta_1 = 1 - \frac{3}{2}e - \frac{27}{160}e^2 - \frac{48511}{25600}e^3 - \frac{1876167}{128000}e^4, \quad \beta_2 = 1 - \frac{3}{2}e + \frac{3}{160}e^2 - \frac{27191}{25600}e^3 - \frac{1291167}{128000}e^4$$

We will now consider the equality $d=0$. According to the last of the equations of (6.10), the coefficient $d^{(4)}$ in the expansion of d in series vanishes only if $\mu_1 = 0$. Calculations show that when $\mu_1 = 0$ we have $d^{(5)} = 0$, and the coefficients $d^{(6)}$, $d^{(7)}$, $d^{(8)}$ are given by the equalities

$$d^{(6)} = \frac{81}{12800}(40\mu_2 + 17)(40\mu_2 - 3), \quad d^{(7)} = \frac{81}{160}(40\mu_2 + 7)\mu_3$$

$$d^{(8)} = \frac{81}{160}(40\mu_2 + 7)\mu_4 + \frac{81}{8}\mu_3^2 + \frac{9}{16}\mu_2^4 + \frac{5049}{320}\mu_2^3 + \frac{761121}{25600}\mu_2^2 - \frac{4033953}{512000}\mu_2 - \frac{162415233}{20480000}$$

The system of equations $d^{(6)} = 0$, $d^{(7)} = 0$, $d^{(8)} = 0$ has two solutions, which correspond to two boundaries of the stability and instability regions (on which $d=0$). Apart from terms of the fourth power of e inclusive, these boundaries are given by the equations

$$\beta_5 = 1 - \frac{17}{40}e^2 - \frac{46033}{576000}e^4, \quad \beta_6 = 1 + \frac{3}{40}e^2 + \frac{13191}{8000}e^4$$

In Fig. 3b in the e, β plane we show six of the curves obtained, which separate the stability and instability regions in the neighbourhood of the point (0,1) of the multiple resonance considered. The instability regions are shown hatched.

6.2. A satellite whose ellipsoid of inertia is close to a sphere ($A \simeq C$)

If the central ellipsoid of inertia of the satellite is a sphere, we have $\alpha = 1$. When $\alpha = 1$ it follows from the equation of the frequencies (6.5) that $\omega_1 = |\beta - 1|$, $\omega_2 = 1$ if $\beta \le 0$ or $\beta \ge 2$ and $\omega_1 = 1$, and $\omega_2 = |\beta - 1|$ if $0 \le \beta \le 2$. In a low-eccentric orbit a multiple parametric resonance is possible in the neighbourhood of points of the axis $\alpha = 1$, in which the quantity 2β is an integer. We will always consider three cases of multiple resonance, which illustrate the algorithms described in Sections 4.2, 3.4 and 3.3.

The resonance $\omega_1 = \omega_2 = 1$. For small values of e we will consider the neighbourhood of the point $P_3(1, 2)$ (Fig. 1). To find the surfaces which separate stability and instability regions in α, β, e space, we put

$$\alpha = 1 + e\nu_1, \quad \beta = 2 + e\mu_1 + e^2\mu_2 + e^3\mu_3 + e^4\mu_4 + \dots \; (\nu_1 \ne 0) \tag{6.11}$$

We substitute the quantities (6.11) into Hamiltonian (6.1), expand in series in powers of e and make the univalent canonical replacement of variables

$$q_1 = x_1, \quad q_2 = x_2, \quad p_1 = -x_2 + X_1, \quad p_2 = -x_1 + X_2$$

which reduce the unperturbed Hamiltonian

$$F_0 = q_1^2 + q_1p_2 + q_2^2 + q_2p_1 + \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$$

to the normal form

$$H_0 = \frac{1}{2}(x_1^2 + X_1^2) + \frac{1}{2}(x_2^2 + X_2^2)$$

Further calculations were carried out using the algorithm of Section 4.2. We obtain the following expression for the coefficient a of the characteristic Eq. (1.4)

$$a = \frac{1}{16}[(4\mu_1 + 11\nu_1)^2 + 27\nu_1^2]e^2 + O(e^3)$$

Since $a > 0$, then, according to conditions (1.6), the boundaries of the stability and instability regions can be found from the relations $b = 0$ or $d = 0$.

Calculations show that the equality $b = 0$ defines two sets of expressions for the coefficients of the expansion (6.11) of the quantity β in series in terms of the quantity ν_1

$$\mu_1 = -2\nu_1, \quad \mu_2 = 2\nu_1^2, \quad \mu_3 = -2\nu_1^3 - 3\nu_1, \quad \mu_4 = 2\nu_1^4$$

and

$$\mu_1 = -2\nu_1, \quad \mu_2 = 2\nu_1^2, \quad \mu_3 = -2\nu_1^3 - 9\nu_1, \quad \mu_4 = 2\nu_1^2(\nu_1^2 + 12)$$

Consequently, the equality $b = 0$ defines two surfaces which separate the stability and instability regions in α, β, e space. In parametric form (the quantities ν_1 and e play the role of the parameters) the equations of the boundary surfaces are given by equalities (6.11), in which the quantities $\mu_1, \mu_2, \dots$ are expressed in terms of ν_1. Eliminating the parameter ν_1 from these equations, we can obtain the equations of the boundary surfaces in explicit form $\beta = \beta_i(\alpha, e)$ $(i = 1, 2)$. Apart from terms of the fourth power inclusive in e and $(\alpha - 1)$, we obtain

$$\beta_1 = 2 - (2 + 3e^2)(\alpha - 1) + 2(\alpha - 1)^2 - 2(\alpha - 1)^3 + 2(\alpha - 1)^4$$

$$\beta_2 = 2 - (2 + 9e^2)(\alpha - 1) + (2 + 24e^2)(\alpha - 1)^2 - 2(\alpha - 1)^3 + 2(\alpha - 1)^4$$

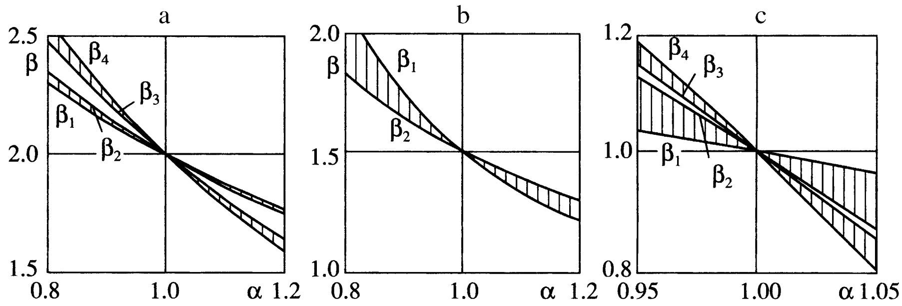

Fig. 4.

From the equality $d=0$ we can similarly obtain two boundary surfaces $\beta=\beta_i(\alpha, e)$ $(i=3, 4)$

$$\beta_3 = 2+(-3.5+4.6443e^2)(\alpha-1)+(3.5-4.9459e^2)(\alpha-1)^2-$$
$$-3.5(\alpha-1)^3+3.8164(\alpha-1)^4$$

$$\beta_4 = 2+(-3.5-9.1443e^2)(\alpha-1)+(3.5+25.9459e^2)(\alpha-1)^2-$$
$$-3.5(\alpha-1)^3+3.8164(\alpha-1)^4$$

To illustrate the above investigation, in Fig. 4a we show a section of the stability and instability regions in the neighbourhood of the point P_3 of the $e=0.2$ plane. The instability regions are shown hatched.

The resonance $\omega_1=\frac{1}{2}$ and $\omega_2=0$. Consider the neighbourhood of the point $P_4(1, \frac{3}{2})$. We will represent the quantities α and β by expansions of the form (6.11) (replacing the number 2 by the number 3/2 in the expansion of β), substitute into the Hamilton function (6.1) and expand it in series of powers of e. The unperturbed Hamiltonian has the form

$$F_0 = \frac{3}{8}q_1^2+\frac{1}{2}q_1p_2+\frac{3}{4}q_2^2+q_2p_1+\frac{1}{2}p_1^2+\frac{1}{2}p_2^2$$

while the frequencies of small oscillations are equal to 1 and ½.

We make successively two canonical univalent replacements of variables using the formulae

$$q_1 = \frac{\sqrt{6}}{3}(\tilde{x}_1-\tilde{x}_2), \quad q_2 = \frac{\sqrt{6}}{3}(\tilde{X}_1+\tilde{X}_2), \quad p_1 = -\frac{\sqrt{6}}{2}\tilde{X}_2, \quad p_2 = -\frac{\sqrt{6}}{2}\tilde{x}_1 \tag{6.12}$$

$$\tilde{x}_1 = \cos\nu x_2+\sin\nu X_2, \quad \tilde{x}_2 = x_1; \quad \tilde{X}_1 = -\sin\nu x_2+\cos\nu X_2, \quad \tilde{X}_2 = X_1 \tag{6.13}$$

Replacement (6.12) reduces F_0 to the sum of Hamiltonians of two harmonic oscillators with frequencies 1 and ½

$$H_0 = \frac{1}{2}(\tilde{x}_1^2+\tilde{X}_1^2)+\frac{1}{4}(\tilde{x}_2^2+\tilde{X}_2^2)$$

while replacement (6.13) eliminates from it the part corresponding to the oscillator with a frequency of unity. (In addition, this replacement leads to renumbering of the oscillators.)

After the replacement of variables (6.12) and (6.13) the unperturbed Hamiltonian takes the form

$$H_0 = \frac{1}{4}(x_1^2+X_1^2)$$

The new frequencies $\omega_1=\frac{1}{2}$ and $\omega_2=0$ correspond to this Hamiltonian, i.e. they formally lead to the case of multiple resonance from Section 3.4.

The Hamiltonian (1.2), converted using the Deprit-Hori method, is given by equality (3.3) (in which $\delta_2=0$). Calculations showed that there is an equation $d_2=-e^2\nu_1^2+O(e^3)$ for the quantity d_2 from formulae (3.4). Since, for small e, the quantity d_2 is negative, the boundaries of the stability and instability regions are given solely by the equality

$d_1 = 0$. As calculations showed, this equality defines two boundary surfaces in α, β, e space. Neglecting quantities, the power of which in e and $(\alpha - 1)$ is greater than the fourth, the equations of these surfaces can be written in the form

$$\beta_1(\alpha, e) = \frac{3}{2} - \left(\frac{5}{2} + \frac{7}{2}e + \frac{3}{2}e^2 - \frac{123}{16}e^3\right)(\alpha - 1) + \left(\frac{11}{2} + \frac{25}{2}e - \frac{339}{40}e^2\right)(\alpha - 1)^2 -$$
$$-\left(\frac{23}{2} + \frac{31}{2}e\right)(\alpha - 1)^3 + \frac{29}{2}(\alpha - 1)^4, \quad \beta_2(\alpha, e) = \beta_1(\alpha, -e)$$

In Fig. 4b we show a section of the stability and instability regions in the neighbourhood of the point P_4 in the plane $e = 0.2$. The instability regions are shown hatched.

The resonance $\omega_1 = 1$ and $\omega_2 = 0$. We will investigate the neighbourhood of the point $P_5(1, 1)$. When $e = 0$, two instability regions adjoin this point (see Fig. 1). One of the regions (for which $\alpha < 1$) is given by the inequalities $\alpha\beta - 1 > 0$, $\alpha\beta + 3\alpha - 4 < 0$ (see relations (6.2) and (6.3)), while the other (for which $\alpha > 1$) is given by the inequalities $\alpha\beta - 1 < 0$, $\alpha\beta + 3\alpha - 4 > 0$. The curves $\omega_1 + \omega_2 = 1$ and $\omega_1 - \omega_2 = 1$, passing into the regions G_1 and G_2 (Fig. 1) respectively, also adjoin the point P_5 (see[11]). These curves are not shown in Fig. 1.

To investigate the stability for small e we will represent α and β by expansions of the form (6.11) (replacing the number 2 by 1 in the expansion of β). The unperturbed Hamiltonian F_0 will be

$$F_0 = \frac{1}{2}q_2^2 + q_2 p_1 + \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$$

The canonical transformation

$$q_1 = x_2 - X_1, \quad q_2 = x_1 - X_2, \quad p_1 = X_2, \; p_2 = X_1$$

converts the function F_0 into its normal form

$$H_0 = \frac{1}{2}(x_1^2 + X_1^2)$$

i.e. we have the case of multiple resonance from Section 3.3.

Following the algorithm of Sections 3.3 and 2.4, we can reduce the perturbed Hamiltonian to the form (2.5) (where $H_0 = 0$). We obtain the following estimates for the coefficients a and b of Eq. (1.4)

$$a = \frac{1}{4}(2\mu_1 + 5\nu_1)^2 e^2 + O(e^3), \quad b = \frac{9}{4}(\mu_1 + \nu_1)(\mu_1 + 4\nu_1)\nu_1^2 e^4 + O(e^5)$$

Hence it also follows from relations (1.6) that the boundaries of the stability and instability regions in α, β, e space are found from the equations $b = 0$ and $d = 0$.

Calculations showed that the equation $b = 0$ defines two boundary surfaces, the parametric equations of which have the form

$$\alpha = 1 + e\nu_1, \quad \beta_1 = 1 - e\nu_1 + e^2\nu_1^2 - e^3\nu_1^3 + e^4\nu_1^4 + \dots \tag{6.14}$$

and

$$\alpha = 1 + e\nu_1, \quad \beta_2 = 1 - 4e\nu_1 + 4e^2\nu_1^2 - e^3\nu_1(4\nu_1^2 - 3) + e^4\nu_1^2(4\nu_1^2 - 21) + \dots \tag{6.15}$$

With an error, the order of which in e and $(\alpha - 1)$ is no less than the fifth, Eqs. (6.14) and (6.15) of the boundary surfaces can be written in the following explicit form

$$\alpha\beta_1 - 1 = 0 \text{ and } \alpha\beta_2 + 3\alpha - 4 + 3e^2\alpha(\alpha - 1)(8 - 7\alpha) = 0 \tag{6.16}$$

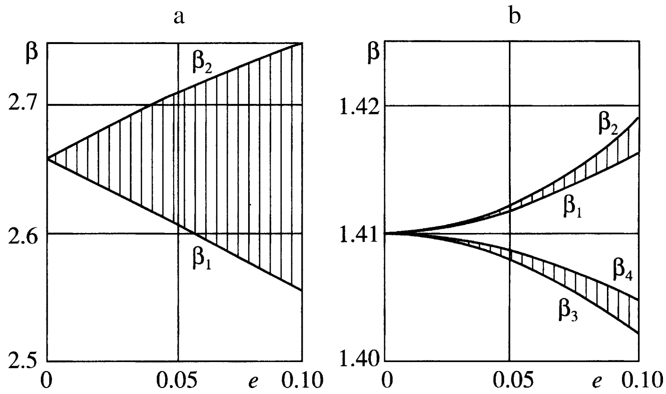

Fig. 5.

The equation $d=0$ also gives two boundary surfaces. Their equations $\beta_3(\alpha, e)$ and $\beta_4(\alpha, e)$ can be written with the same accuracy as follows

$$\beta_3(\alpha, e) = 1-(4.6213-3.0133e+3.0455e^2-8.3017e^3)(\alpha-1)-(0.0152-19.7160e+ \\ +52.5933e^2)(\alpha-1)^2-(16.9242-26.8478e)(\alpha-1)^3-48.9268(\alpha-1)^4 \\ \beta_4(\alpha, e) = \beta_3(\alpha, -e) \tag{6.17}$$

In Fig. 4c we show a section of the stability and instability regions in the neighbourhood of the point P_5 in the plane $e=0.2$. The instability regions are shown hatched. The instability regions lying between curves β_1 and β_2 are somewhat contracted instability regions for small e, which exist in a circular orbit, while the instability regions which lie between the curves β_3 and β_4, arise from the curves $\omega_1+\omega_2=1$ and $\omega_1-\omega_2=1$ for small e.

6.3. *Some other examples of multiple resonances*

The resonance which occurs when one of the numbers $2\omega_1$ or $2\omega_2$ is even, while the other is odd. Consider the neighbourhoods of the two points $P_6(3/4, 8/3)$ and $P_7(17/12, 24/17)$ from region G_1 in Fig. 1. When $e=0$, at the point P_6 we have $\omega_1=1$ and $\omega_2=\frac{1}{2}$, while at the point P_7 we have $\omega_1=3/2$ and $\omega_2=1$. For each of these points we fix the parameter α and, for small e, using the algorithm of Section 5, we construct stability and instability regions in the e, β plane.

The Hamiltonian, converted using the Deprit-Hori method, is reduced to the form (3.3), when the equations for the variables y_1, Y_1 and y_2, Y_2 are separated. From the conditions $d_1>0$ and $d_2>0$ (see Eqs. (3.4) with $\delta_2=0$) for each of the two points considered we obtain two instability regions.

It turned out that for the point P_6 one of the instability regions lies inside the other. Finally we obtain an instability region $\beta_1<\beta<\beta_2$, where $\beta_1(e)$ and $\beta_2(e)$, up to terms of the fourth power in e inclusive, are given by the equations

$$\beta_1(e) = \frac{8}{3}-\frac{3}{2}e-\frac{301}{160}e^2+\frac{117047}{12800}e^3-\frac{17349077}{1433600}e^4, \quad \beta_2(e) = \beta_1(-e)$$

In Fig. 5a the instability region is shown hatched.

In the neighbourhood of the point P_7 the two instability regions obtained from the conditions $d_1>0$ and $d_2>0$, do not intersect. They are given, up to the fourth power of e inclusive, by the inequalities $\beta_1<\beta<\beta_2$ and $\beta_3<\beta<\beta_4$, where

$$\beta_1(e) = \frac{24}{17}+\frac{17685}{7616}e^2-\frac{40285}{8704}e^3-\frac{693927925}{67436544}e^4, \quad \beta_2(e) = \beta_1(-e)$$

$$\beta_3(e) = \frac{24}{17}-\frac{820}{357}e^2+\frac{20267165}{1416933}e^4, \quad \beta_4(e) = \frac{24}{17}-\frac{180}{119}e^2+\frac{610475}{52479}e^4$$

These instability regions are shown hatched in Fig. 5b.

The resonance $2\omega_1 = n_1$ and $\omega_2 = 0$ (n_1 is an odd number). We will consider once again the two points $P_8(3/4, 4/3)$ and $P_9(17/12, 12/17)$, in the neighbourhood of which the Hamiltonian, converted using Deprit-Hori method, has the form (3.3), which allows of separations of the variables. As in the previous section, for each of these points we fix the parameter α and construct stability and instability regions in the e, β plane.

Both of the points considered lie on the boundaries of the stability regions of the steady rotation of the satellite being considered: P_8 is on the boundary of the region G_2 and P_9 is on the boundary of the region G_1 (Fig. 1).

For the point P_8 the unperturbed Hamiltonian F_0 has the form

$$F_0 = -\frac{3}{8}q_1^2 + \frac{1}{2}q_2^2 + q_2p_1 + \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$$

In the unperturbed system with this Hamiltonian $\omega_1 = \frac{1}{2}$ and $\omega_2 = 0$. The canonical univalent replacement of variables

$$q_1 = -\sqrt{2}X_1 - \frac{4\sqrt{3}}{3}X_2,\ \ q_2 = 2\sqrt{2}x_1 - \sqrt{3}x_2,\ \ p_1 = -\frac{3\sqrt{2}}{2}x_1 + \sqrt{3}x_2,\ \ p_2 = \sqrt{2}X_1 + \sqrt{3}X_2$$

reduces F_0 to the normal form

$$H_0 = \frac{1}{4}(x_1^2 + X_1^2) - \frac{1}{2}X_2^2$$

For the point P_9 we have the following unperturbed Hamiltonian

$$F_0 = \frac{5}{8}q_1^2 + \frac{1}{2}q_2^2 + q_2p_1 + \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$$

The frequencies $\omega_1 = 3/2$ and $\omega_2 = 0$ correspond to this. Using the canonical univalent replacement of variables

$$q_1 = -\frac{\sqrt{6}}{3}X_1 + \frac{4\sqrt{5}}{15}X_2,\ \ q_2 = \frac{2\sqrt{6}}{9}x_1 + \frac{\sqrt{5}}{3}x_2,\ \ p_1 = \frac{5\sqrt{6}}{18}x_1 - \frac{\sqrt{5}}{3}x_2,\ \ p_2 = \frac{\sqrt{6}}{3}X_1 + \frac{\sqrt{5}}{3}X_2$$

the function F_0 can be reduced to the normal form

$$H_0 = \frac{3}{4}(x_1^2 + X_1^2) + \frac{1}{2}X_2^2$$

Hence, both of the cases considered correspond to the multiple resonances from Section 3.2. In this case we must put $\delta_2 = -1$ and $\delta_2 = 1$ in relations (3.3) and (3.4) for the points P_8 and P_9 respectively.

In the neighbourhood of the point P_8, using the algorithm of Section 3.2, from the conditions $d_1 > 0$ and $d_2 > 0$ we can obtain two instability regions, $\beta_1 < \beta < \beta_2$ and $\beta > \beta_3$. The functions $\beta_i(e)$ ($i = 1, 2, 3$) are given by the following equations, up to terms of the order of e^4 inclusive

$$\beta_1(e) = \frac{4}{3} - \frac{5}{6}e + \frac{127}{96}e^2 - \frac{4597}{4608}e^3 + \frac{2272399}{1843200}e^4,\quad \beta_2(e) = \beta_1(-e),\quad \beta_3(e) = \frac{4}{3} + \frac{9}{40}e^4$$

For small e these instability regions intersect, as a result of which the instability region in the neighbourhood of the point P_8 is given by a single equation $\beta > \beta_1$. This region is shown hatched in Fig. 6a. When $\beta < \beta_1$ we have stability.

In the neighbourhood of the point P_9 we can similarly obtain two instability regions, $\beta_1 < \beta < \beta_2$ and $\beta < \beta_3$, where

$$\beta_1(e) = \frac{12}{17} + \frac{2673}{1088}e^2 - \frac{7545}{8704}e^3 + \frac{10299697641}{272957440}e^4,\ \ \beta_2(e) = \beta_1(-e),\ \ \beta_3(e) = \frac{12}{17} - \frac{405}{952}e^4$$

For small e these regions do not intersect. They are shown hatched in Fig. 6b.

The resonance $\omega_1 = \omega_2 = 0$. We will investigate the stability of the steady rotation of a satellite for values of the parameters α and β lying in a small neighbourhood of the point $P_{10}(2/3, 3/2)$ (Fig. 1). We will put

$$\alpha = \frac{2}{3} + e^2\nu_2,\quad \alpha\beta = 1 + e^2\mu_2 + e^3\mu_3 + e^4\mu_4 + e^5\mu_5 + \dots \tag{6.18}$$

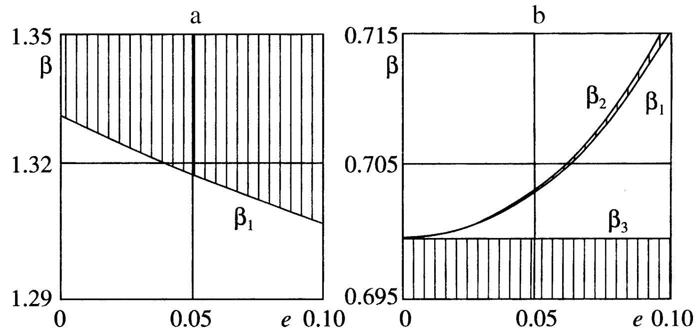

Fig. 6.

The unperturbed Hamiltonian has the form

$$F_0 = -\frac{1}{2}q_1^2 + \frac{1}{2}q_2^2 + q_2 p_1 + \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$$

Zero frequencies of small oscillations $\omega_1 = \omega_2 = 0$ correspond to this. The canonical univalent replacement of variables

$$q_1 = -x_1 - \frac{1}{2}x_2, \quad q_2 = -\frac{1}{2}X_1 + X_2, \quad p_1 = -\frac{1}{2}X_1 - X_2, \quad p_2 = x_1 - \frac{1}{2}x_2$$

reduces the function F_0 to its normal form

$$H_0 = \frac{1}{2}X_1^2 - x_1 x_2$$

Consequently, we have the multiple parametric resonance considered in Section 2.1.

Calculations using the algorithm of Section 2.1 showed that, for the coefficients a and b of characteristic Eq. (1.4) and the quantity d from relations (1.5) we have the following expressions

$$a = 3(3+\nu_2)e^2 + O(e^4), \quad b = -e^2\mu_2 - e^3\mu_3 - \left[\mu_4 - \mu_2^2 + \mu_2\left(\frac{65}{2} - 3\nu_2\right) - \frac{27}{128}\right]e^4 -$$

$$-\left[\mu_5 - \mu_3\left(3\nu_2 + 2\mu_2 - \frac{65}{2}\right)\right]e^5 + O(e^6)$$

$$d = 4e^2\mu_2 + 4e^3\mu_3 + \left[4\mu_4 - 4\mu_2^2 + 2\mu_2(65 - 6\nu_2) + (3\nu_2 + 9)^2 - \frac{27}{32}\right]e^4 +$$

$$+ [4\mu_5 - 2\mu_3(4\mu_2 + 6\nu_2 - 65)]e^5 + O(e^6)$$

The stability and instability regions are found from relations (1.5). For small values of e, from the equations $b=0$ and $d=0$ we obtain $\mu_1 = \mu_2 = \mu_5 = 0$, while $\mu_4 = 27/128$ (in the case $b=0$) and $\mu_4 = 27/128 - (3\nu_2+9)^2/4$ (in the case $d=0$). Hence it also follows from expansions (6.18) that in α, β, e parameter space the quantities b and d vanish on the surfaces $f_b = 0$ and $f_d = 0$, where the functions f_b and f_d can be written, with an error of the order of e^6, in the form

$$f_b = s_1 - \frac{27}{128}e^4, \quad f_d = \Delta + 54\left(\alpha - \frac{2}{3}\right)e^2 + \frac{2565}{32}e^4$$

Here s_1 and Δ are quantities defined by Eqs. (6.2) and (6.4).

At common points $P_*(\alpha_*, \beta_*)$ of the surfaces $f_b = 0, f_d = 0$ we have

$$\alpha_* = \frac{2}{3} - 3e^2 + O(e^4), \quad \beta_* = \frac{3}{2} + \frac{27}{4}e^2 + O(e^4) \tag{6.19}$$

At these points both coefficients a and b of characteristic Eq. (1.4) vanish.

If the quantity e is sufficiently small, then when $\alpha > \alpha_*$ the coefficient a is positive, and the stability region is given by the system of inequalities

$$f_b < 0, \quad f_d > 0 \tag{6.20}$$

Outside this region there is instability.

When $e = 0$, inequalities (6.2) become the inequalities $s_1 < 0$, $\Delta > 0$, which specify (see relations (6.3)) the part of the stability region G_2 of steady rotation of the satellite in a circular orbit in the neighbourhood of the point P_{10} (Fig. 1). For a small fixed value of e the boundaries of this part of the region G_2 are deformed, and the point P_{10} becomes the point P with coordinates (6.19).

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